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Empty space–times with separable Hamilton–Jacobi equation

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Abstract. All empty space–times admitting a one-parameter group of motions and in which the Hamilton–Jacobi equation is (partially) separable are obtained. Several different cases of such empty space–times exist and the Riemann tensor is found to be either type D or N. The results presented here complete the search for empty space–times with separable Hamilton–Jacobi equation.

1. Introduction

The problem of solving the Hamilton–Jacobi equation by separation of variables has been studied extensively in the past (Liouville 1846, Stäckel 1893). More recently a great deal of work has been carried out on the subject of the separability of the Hamilton–Jacobi equation in Riemannian spaces and, in particular, in the curved space–times of general relativity. Amongst the motivations for this work has been the need to study the behaviour of geodesics in order to find global properties of the underlying manifold and also the need to give covariant criteria for separability.

The first of the more recent papers on the subject is written by Carter (1968) who studied space–times admitting a two-parameter Abelian group of motions and in which the Hamilton–Jacobi equation is separable. Carter solved the empty-space field equations, for such space–times, with and without the cosmological term and also the Einstein–Maxwell equations. In order to solve these equations completely Carter assumed the separability of the Klein–Gordon equation so obtaining further restrictions on the form of the metric tensor. However, it is now known (Carter, private communication) that in empty space–times separability of the Klein–Gordon equation follows directly from the separability of the Hamilton–Jacobi equation.

Later Woodhouse (1975) has shown that if a separable coordinate exists then it is adapted either to a Killing vector or to an eigenvector of a Killing tensor. He applies his results to space–times of Petrov type D. These results have been extended by Dietz (1976) to the case of separability of the Klein–Gordon equation.

In the present paper the authors review the different possible cases of separability which arise for the Hamilton–Jacobi equation. It appears that the only case for which the empty-space solutions have not yet been found is the case when one coordinate is ignorable (i.e. adapted to a motion of the space–time) and one coordinate is separable. This case is studied in detail and the empty-space field equations solved. Unfortunately the solutions found are either of Petrov type D or of Petrov type N with non-diverging rays. They are therefore not new and are contained in either the type-D solutions of Kinnersley (1969) or the plane-fronted waves of Kundt (1961).

2. Classification of the different cases of separability and the corresponding canonical forms for the metric

Let (M_4, g) be a four-dimensional pseudo-Riemannian manifold with Lorentz signature (i.e. a space–time). A set of local coordinates will be denoted by x^α, $\alpha = 1, 2, 3, 4$.

The Hamilton–Jacobi equation for a geodesic is

$$g^{\alpha\beta}S_{,\alpha}S_{,\beta} - m^2 = 0 \tag{2.1}$$

where $g^{\alpha\beta}$ are the contravariant components of the metric tensor, the comma denotes a partial derivative, and the constant m^2 satisfies $m^2 = 0$, $m^2 > 0$ and $m^2 < 0$ for null, time-like and space-like geodesics respectively. Canonical forms for $g^{\alpha\beta}$ have been constructed by Dietz (1976) under the hypothesis that the Hamilton–Jacobi equation, after multiplication by an integrating factor U, can be solved by (partial) separation of variables. The various canonical forms depend upon the number of separable coordinates and the number of ignorable coordinates admitted by the metric (i.e. coordinates adapted to an Abelian group of motions). In fact the ignorable coordinates appear linearly with constant coefficients in the solution S of the Hamilton–Jacobi equation and so the functional form of S can be used to specify both the form of the separation and the ignorable coordinates which are assumed to exist. For example

$$S = S_1(x^1) + S_2(x^2, x^3) + kx^4$$

is the case in which x^4 is an ignorable coordinate and x^1 separates completely. Notice that the ignorable coordinates are trivially separable and the term 'separable coordinate' will be used here to refer to non-trivial separable coordinates only.

It is found that in all cases the integrating factor U separates in exactly the same way as S (the ignorable coordinates do not, however, appear in U so that, for example, in the above case $U = U_1(x^1) + U_2(x^2, x^3)$). Since U appears only as a conformal factor in the metric it is convenient to introduce a conformal metric $\tilde{g}^{\alpha\beta}$ by

$$\tilde{g}^{\alpha\beta} = Ug^{\alpha\beta}. \tag{2.2}$$

The various canonical forms for the conformal metric $\tilde{g}^{\alpha\beta}$ and the coordinate transformations leaving these forms invariant are summarized in table 1. In table 1 indices a, b take the values 2, 3 and 4, indices i, j take the values 3 and 4, and indices A, B take the values 1, 2. The arrows in table 1 indicate when one entry is a special case of the preceding entry and the ϵ are all equal to ± 1 excepting in the fourth and fifth cases when ϵ_1 can take the value zero corresponding to a null separable coordinate. Notice that $S = S_1(x^1) + S_2(x^2) + S_3(x^3, x^4)$ is also a special case of $S = S_1(x^1, x^2) + S_3(x^3, x^4)$.

From table 1 it can be seen that if no ignorable coordinate exists then $\tilde{g}^{\alpha\beta}$ is reducible so that the space–time metric itself, i.e. $g^{\alpha\beta}$, is conformally reducible. Such space–times have been discussed by Petrov (1969) and the corresponding Einstein empty-space field equations solved. The case $S = S_1(x^1) + S_2(x^2) + kx^3 + k'x^4$ is the one discussed by Carter (1968) and the case $S = S_1(x^1) + kx^2 + k'x^3 + k''x^4$ leads to the space–time metrics discussed by Dautcourt *et al* (1961). This leaves the two cases in which one and only one ignorable coordinate occurs. Since $S = S_1(x^1) + S_2(x^2) + S_3(x^3) + kx^4$ is a special case of $S = S_1(x^1) + S_2(x^2, x^3) + kx^4$ attention will now be confined to this latter case.

When $\epsilon_1 = \pm 1$ the allowable transformations can be used, with a suitable choice of the functions $x_1^{4\prime}(x^1)$ and $x_2^{4\prime}(x^2, x^3)$, to set g^{14} and g^{24} equal to zero and the general

Table 1. Canonical forms for the different cases of separability.

S	$\tilde{g}^{\alpha\beta}$	Allowable transformations
$S_1(x^1)+S_2(x^2,x^3,x^4)$ ↓	$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g^{ab}(x^2,x^3,x^4) & \\ 0 & & & \end{pmatrix}$	$x^{1'}=x^1+\text{constant}$ $x^{a'}=x^{a'}(x^2,x^3,x^4)$
$S_1(x^1)+S_2(x^2)+S_3(x^3,x^4)$ ↓	$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & g^{ij}(x^3,x^4) \end{pmatrix}$	$x^{1'}=x^1+\text{constant}$ $x^{2'}=x^2+\text{constant}$ $x^{i'}=x^{i'}(x^3,x^4)$
$S_1(x^1)+S_2(x^2)+S_3(x^3)+S_4(x^4)$	$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{pmatrix}$	$x^{1'}=x^1+\text{constant}$ $x^{2'}=x^2+\text{constant}$ $x^{3'}=x^3+\text{constant}$ $x^{4'}=x^4+\text{constant}$
$S_1(x^1)+S_2(x^2,x^3)+kx^4$ ↓	$\begin{pmatrix} \epsilon_1 & 0 & 0 & g^{14}(x^1) \\ 0 & & & \\ 0 & & g_2^{ab}(x^2,x^3)+\delta_4^a\delta_4^b f_1^4(x^1) & \\ g^{14}(x^1) & & & \end{pmatrix}$	$x^{1'}=\begin{cases} x^1+\text{constant} & \text{if } \epsilon_1=\pm 1 \\ x^{1'}(x^1) & \text{if } \epsilon_1=0 \end{cases}$ $x^{2'}=x^{2'}(x^2,x^3)$ $x^{3'}=x^{3'}(x^2,x^3)$ $x^{4'}=x_1^{4'}(x^1)+x_2^{4'}(x^2,x^3)+cx^4$
$S_1(x^1)+S_2(x^2)+S_3(x^3)+kx^4$	$\begin{pmatrix} \epsilon_1 & 0 & 0 & g^{14}(x^1) \\ 0 & \epsilon_2 & 0 & g^{24}(x^2) \\ 0 & 0 & \epsilon_3 & g^{34}(x^3) \\ g^{14}(x^1) & g^{24}(x^2) & g^{34}(x^3) & g^{44} \end{pmatrix}$ with $g^{44}=g_1^{44}(x^1)+g_2^{44}(x^2)+g_3^{44}(x^3)$	$x^{1'}=x^1+\text{constant}$ $x^{2'}=x^2+\text{constant}$ $x^{3'}=x^3+\text{constant}$ $x^{4'}=x_1^{4'}(x^1)+x_2^{4'}(x^2)+x_3^{4'}(x^3)+cx^4$
$S_1(x^1)+S_2(x^2)+kx^3+k'x^4$	$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & g_1^{ij}(x^1)+g_2^{ij}(x^2) & \\ 0 & 0 & & \end{pmatrix}$	$x^{1'}=x^1+\text{constant}$ $x^{2'}=x^2+\text{constant}$ $x^{3'}=x_1^{3'}(x^1)+x_2^{3'}(x^2)+cx^3$ $x^{4'}=x_1^{4'}(x')+x_2^{4'}(x^2)+c'x^4$
$S_1(x^1)+kx^2+k'x^3+k''x^4$	$(\tilde{g}^{\alpha\beta}(x^1))$	$x^{1'}=x^{1'}(x^1)$ $x^{2'}=x_1^{2'}(x^1)+cx^2$ $x^{3'}=x_1^{3'}(x^1)+c'x^3$ $x^{4'}=x_1^{4'}(x^1)+c''x^4$
$S_1(x^1,x^2)+S_3(x^3,x^4)$†	$\begin{pmatrix} g^{AB}(x^1,x^2) & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & g^{ij}(x^3,x^4) \end{pmatrix}$	$x^{A'}=x^{A'}(x^1,x^2)$ $x^{i'}=x^{i'}(x^3,x^4)$

† This case has not been discussed elsewhere.

transformation of x^2 and x^3 can be used to introduce a geodesic coordinate system into the two-space

$$\begin{pmatrix} g^{22} & g^{23} \\ g^{23} & g^{33} \end{pmatrix}.$$

In order to fix ϵ_1 it will now be assumed that the separable coordinate x^1 is time-like so that $\epsilon_1 = +1$. Re-defining the functions the final form of the metric $g^{\alpha\beta}$ which will be used in order to solve the Einstein field equations is

$$U^{-1}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2A & 2B \\ 0 & 0 & 2B & -2F \end{pmatrix} \tag{2.3}$$

where

$$U = U_1(x^1) + \underset{\sim}{U}(x^2, x^3), \qquad F = F_1(x^1) + \underset{\sim}{F}(x^2, x^3)$$
$$A = A(x^2 x^3), \qquad B = B(x^2, x^3),$$

and the factors 2 have been introduced for future convenience. This final form of the metric is invariant under the coordinate transformations

$$\left.\begin{aligned} x^{1'} &= x^1 + \text{constant} \\ x^{2'} &= x^2 + \text{constant} \\ x^{3'} &= x^{3'}(x^3) \\ x^{4'} &= \text{constant} + x_3^{4'}(x^3) + cx^4 \end{aligned}\right\} \quad \text{if } A \neq 0 \tag{2.4a}$$

or

$$\left.\begin{aligned} x^{1'} &= x^1 + \text{constant} \\ x^{2'} &= x^2 + x^{2'}(x^3) \\ x^{3'} &= x^{3'}(x^3) \\ x^{4'} &= \text{constant} + x_3^{4'}(x^3) + cx^4. \end{aligned}\right\} \quad \text{if } A = 0 \tag{2.4b}$$

The case when the separable coordinate x^1 is space-like can be inferred from the case discussed here by a complex transformation. In fact in certain of the solutions obtained the coordinate is necessarily space-like and a complex transformation has to be carried out before a space–time with the appropriate signature is obtained. Notice that if $F_1(x^1)$ is a constant then the metric becomes a special case of the metric corresponding to $S = S_1(x^1) + S_2(x^2, x^3, x^4)$. For this reason it will be assumed throughout the rest of this paper that $F_1(x^1)$ is non-constant.

When $\epsilon_1 = 0$, that is when the separable coordinate x^1 is null, the allowable transformations can be used to reduce the metric $g^{\alpha\beta}$ to the form

$$U^{-1}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2A & 2B \\ 1 & 0 & 2B & -2F \end{pmatrix} \tag{2.5}$$

where

$$U = U_1(x^1) + \underset{\sim}{U}(x^2, x^3), \qquad F = F(x^2, x^3),$$
$$A = A(x^2, x^3), \qquad B = B(x^2, x^3).$$

The form of this metric is invariant under the coordinate transformations

$$\begin{aligned} x^{1'} &= x^1/c + \text{constant} \\ x^{2'} &= x^2 + \text{constant} \\ x^{3'} &= x^{3'}(x^3) \\ x^{4'} &= c'x^1 + x_2^{4'}(x^3) + cx^4. \end{aligned} \tag{2.6}$$

3. The type-D solutions

Solution of the Einstein empty-space field equations for the metric (2.3) with $A \neq 0$ yields space–times of Petrov type D. In order to write down the field equations the null tetrad formalism introduced by Newman and Penrose (1962) will be used. The vectors

$$\tilde{l}_\alpha = (\delta^1_\alpha + \delta^2_\alpha)/2, \qquad \tilde{n}_\alpha = (\delta^1_\alpha - \delta^2_\alpha)/2$$

and

$$\tilde{m}_\alpha = F^{1/2} d^{-1/2} [\delta^3_\alpha + F^{-1}(B - d^{1/2}\mathrm{i})\delta^4_\alpha]/2,$$

with $d = AF - B^2$, form a null tetrad for the conformal metric $\tilde{g}^{\alpha\beta}$ associated with the space–time metric $g^{\alpha\beta}$ given by (2.3). The intrinsic derivatives and non-zero spin coefficients for the tetrad are

$$\tilde{\mathrm{D}} = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}, \qquad \tilde{\Delta} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \qquad \tilde{\delta} = -F^{-1/2}(d^{1/2} + \mathrm{i}B)\frac{\partial}{\partial x^3} + \mathrm{i}F^{1/2}\frac{\partial}{\partial x^4}$$

and

$$\begin{aligned}
\tilde{\alpha} &= -\bar{\tilde{\beta}} = -\tfrac{1}{4}F^{-3/2}d^{1/2}\underset{\sim}{F}_{,3} + \tfrac{1}{4}\mathrm{i}F^{-1/2}B(d^{-1}d_{,3} + F^{-1}\underset{\sim}{F}_{,3} - 2B^{-1}B_{,3}),\\
\tilde{\rho} &= \tfrac{1}{4}d^{-1}(AF_{1,1} - d_{,2}), \qquad \tilde{\mu} = -\tfrac{1}{4}d^{-1}(AF_{1,1} + d_{,2}),\\
\tilde{\epsilon} &= -\tfrac{1}{4}\mathrm{i}[Bd^{-1/2}F^{-1}(F_{1,1} - \underset{\sim}{F}_{,2}) + d^{-1/2}B_{,2}],\\
\tilde{\gamma} &= -\tfrac{1}{4}\mathrm{i}[Bd^{-1/2}F^{-1}(F_{1,1} + \underset{\sim}{F}_{,2}) - d^{-1/2}B_{,2}],\\
\tilde{\sigma} &= \tfrac{1}{2}F^{-1}(\underset{\sim}{F}_{,2} - F_{1,1}) + \tfrac{1}{4}d^{-1}(AF_{1,1} - d_{,2}) - \tfrac{1}{2}d^{-1/2}\mathrm{i}[BF^{-1}(F_{1,1} - \underset{\sim}{F}_{,2}) + B_{,2}],\\
\tilde{\lambda} &= \tfrac{1}{2}F^{-1}(\underset{\sim}{F}_{,2} + F_{1,1}) - \tfrac{1}{4}d^{-1}(AF_{1,1} + d_{,2}) - \tfrac{1}{2}d^{-1/2}\mathrm{i}[BF^{-1}(F_{1,1} + \underset{\sim}{F}_{,2}) - B_{,2}].
\end{aligned}$$

Using the Newman–Penrose field equations and the connection, given in the appendix, between the tetrad components of the Ricci tensors of the conformally related metrics $g^{\alpha\beta}$ and $\tilde{g}^{\alpha\beta}$ it is found that two of the Einstein empty-space field equations, namely $\phi_{00} - \phi_{22} = 0$ and $\mathrm{Re}(\phi_{01} + \phi_{21}) = 0$, can be written as

$$F_{1,1}A^2(\underset{\sim}{F} - B^2/A)_{,2} = 2d^2U^{-2}U_{1,1}U_{,2} \tag{3.1}$$

and

$$F_{1,1}A^2(\underset{\sim}{F} - B^2/A)_{,3} = 2d^2U^{-2}U_{1,1}\underset{\sim}{U}_{,3}. \tag{3.2}$$

From equation (3.1) it can be seen that $\ln d - \ln U$ separates into the sum of a function of x^1 and a function of x^2, x^3. Hence $(\ln d - \ln U)_{,12} = 0$. Written out explicitly this condition becomes

$$F_{1,1}A^2(\underset{\sim}{F} - B^2/A)_{,2} = d^2U^{-2}U_{1,1}\underset{\sim}{U}_{,2}.$$

Comparing this with equation (3.1) yields

$$U_{1,1}\underset{\sim}{U}_{,2} = 0 \tag{3.3}$$

and

$$(\underset{\sim}{F} - B^2/A)_{,2} = 0. \tag{3.4}$$

Similarly equation (3.2) yields

$$U_{1,1}\underset{\sim}{U}_{,3} = 0 \tag{3.5}$$

and

$$(\underset{\sim}{F} - B^2/A)_{,3} = 0. \tag{3.6†}$$

Equations (3.4) and (3.6) give

$$\underset{\sim}{F} = B^2/A \tag{3.7}$$

where a constant of integration has been absorbed into the function $F_1(x^1)$. It follows that the determinant d takes the simple form

$$d = AF_1. \tag{3.8}$$

From equations (3.3) and (3.5) it can be seen that two distinct cases arise: case 1, with $U_{1,1} \neq 0$ and $\underset{\sim}{U}_{,2} = \underset{\sim}{U}_{,3} = 0$; case 2, with $U_{1,1} = 0$. By absorbing a constant of integration into $\underset{\sim}{U}_1$ the function $\underset{\sim}{U}$ can be assumed to be zero in case 1. Similarly U_1 can be assumed to be zero in case 2.

The integration of the remaining field equations involves considerable algebraic manipulation and will be omitted here. The forms found, after simplification using the coordinate transformations (2.4*a*), for the various functions appearing in the metric (2.3) are listed below.

Case 1(*a*)

$$\underset{\sim}{U} = 0, \qquad B^2 = A\underset{\sim}{F},$$

$$A = \pm[\sin(k_1x^2 + \beta(x^3))]^{-2} \qquad \text{with } k_1 \neq 0$$

$$\underset{\sim}{F} = 4\frac{k_0^2}{k_1^2}[\cot(k_1x^2 + \beta(x^3))]^2$$

$$U_1 = \begin{cases} \dfrac{k_0^2 + k_3^2}{k_2}[1 + \sin^2(k_1x^1) - 2k_3(k_0^2 + k_3^2)^{-1/2}\sin(k_1x^1)] & \text{if } k_2 \neq 0 \\ k_3 \sin(k_1x^1) & \text{if } k_2 = 0 \end{cases}$$

$$F_1 = (k_2 - k_0^2U_1^{-1})/(U_1^{-1/2})^2_{,1}.$$

† It turns out that equations (3.3)–(3.6) are the further necessary and sufficient condition for the Klein–Gordon equation to separate. Hence, in this case, the empty-space field equations ensure that separability of the Hamilton–Jacobi equation implies separability of the Klein–Gordon equation. This is a special case of a general result found by Carter (private communication).

Case 1(*b*)

$$\underset{\sim}{U}=0, \qquad B^2=A\underset{\sim}{F}$$
$$A=\pm(x^2+\beta(x^3))^{-2}$$
$$\underset{\sim}{F}=k_0^2(x^2+\beta(x^3))^2$$
$$U_1=\begin{cases}\dfrac{k_0^2}{k_2}+(k_3x^1)^4 & \text{if } k_2\neq 0\\ k_3x^1 & \text{if } k_2=0\dagger\end{cases}$$
$$F_1=(k_2-k_0^2U_1^{-1})/(U_1^{-1/2})^2_{,1}.$$

Case 2

$$U_1=0, \qquad B^2=A\underset{\sim}{F},$$
$$F=0,$$
$$F_1=\begin{cases}k_2[\sin(k_1x^1)]^{-2} & \text{if } k_1\neq 0\\ k_2(x^1)^{-2} & \text{if } k_1=0.\end{cases}$$

$\underset{\sim}{U}$ is determined, in terms of elliptic integrals, from the equation

$$(U^{-1/2})^2_{,2}=-\alpha(x^3)-k_1^2U^{-1}+k_3U^{-3/2}$$

and

$$A=\alpha(x^3)/(U^{-1/2})^2_{,3}.$$

In the above the k are arbitrary constants and $\alpha(x^3)$, $\beta(x^3)$ are arbitrary functions. In all cases the only non-vanishing null tetrad components of the Weyl tensor are ψ_0, ψ_2, ψ_4. These satisfy $\psi_0\psi_4=9\psi_2^2$ and so the space-times are of Petrov type D.

4. The plane-fronted waves

Solution of the Einstein empty-space field equations for the metric (2.3) with $A=0$ and for the metric (2.5) yields space-times of type N with non-diverging rays, that is plane-fronted waves. Again the calculations have all been carried out using the null tetrad formalism and the results are summarized below.

Case 1. The metric (2.3) with $A=0$:

$$U=2\,e^{2x^2}$$
$$F=\pm\sin(2x^1)+2\eta(x^3)\,e^{2x^2}(2x^2+k)$$
$$B=e^{2x^2}.$$

Case 2. The metric (2.5):

$$B=0$$
$$A=(x^2+\beta(x^3))^{-2}$$
$$U=\begin{cases}k_2[\sin(k_1x^1)]^{-2} & \text{if } k_1\neq 0\\ k_2(x^1)^{-2} & \text{if } k_1=0.\end{cases}$$

† In cases 1(*a*) and 1(*b*) the metrics with $k_2=0$ do not, in fact, have a normal hyperbolic signature.

F is determined from the equation

$$F_{,22}-\tfrac{1}{2}A^{-1}F_{,2}A_{,2}+2AF_{,33}+A_{,3}F_{,3}-2k_1^2=0.$$

Further details of the calculations can be obtained from the authors.

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Appendix. Relations between conformally related tetrads

Consider two comformally related metrics with

$$g_{\alpha\beta}=U\tilde{g}_{\alpha\beta}.$$

Null tetrads can be chosen for the two metrics related by the following conformal transformations:

$$l_\alpha=U^{1/2}\tilde{l}_\alpha;\qquad n_\alpha=U^{1/2}\tilde{n}_\alpha;\qquad m_\alpha=U^{1/2}\tilde{m}_\alpha,$$

so that

$$\mathrm{D}=U^{-1/2}\tilde{\mathrm{D}},\qquad \Delta=U^{-1/2}\tilde{\Delta},\qquad \delta=U^{-1/2}\tilde{\delta}.$$

Using the commutation relationships the spin coefficients are found to be related as follows:

$$\mu=U^{-1/2}\tilde{\mu}+\tfrac{1}{2}U^{-3/2}\,\tilde{\Delta}U \qquad \rho=U^{-1/2}\tilde{\rho}-\tfrac{1}{2}U^{-3/2}\,\tilde{\mathrm{D}}U$$

$$\gamma=U^{-1/2}\tilde{\gamma}-\tfrac{1}{4}U^{-3/2}\,\tilde{\Delta}U \qquad \epsilon=U^{-1/2}\tilde{\epsilon}+\tfrac{1}{4}U^{-3/2}\,\tilde{\mathrm{D}}U$$

$$\tau=U^{-1/2}\tilde{\tau}-\tfrac{1}{2}U^{-3/2}\,\tilde{\delta}U \qquad \pi=U^{-1/2}\tilde{\pi}+\tfrac{1}{2}U^{-3/2}\bar{\tilde{\delta}}U$$

$$\beta=U^{-1/2}\tilde{\beta}+\tfrac{1}{4}U^{-3/2}\,\tilde{\delta}U \qquad \alpha=U^{-1/2}\tilde{\alpha}-\tfrac{1}{4}U^{-3/2}\,\bar{\tilde{\delta}}U$$

$$\kappa=U^{-1/2}\tilde{\kappa} \qquad \nu=U^{-1/2}\tilde{\nu}$$

$$\sigma=U^{-1/2}\tilde{\sigma} \qquad \lambda=U^{-1/2}\tilde{\lambda}$$

and, using the Newman–Penrose field equations, the tetrad components of the Ricci tensor and the Ricci scalar are found to be related as follows:

$$\phi_{00}=U^{-1}[\tilde{\phi}_{00}-\tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}\tilde{\mathrm{D}}U+\tfrac{3}{4}U^{-2}\,\tilde{\mathrm{D}}U\,\tilde{\mathrm{D}}U-\tfrac{1}{2}U^{-1}\,\tilde{\delta}U\bar{\tilde{\kappa}}-\tfrac{1}{2}U^{-1}\,\bar{\tilde{\delta}}U\,\tilde{\kappa}$$
$$+\tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U(\tilde{\epsilon}+\bar{\tilde{\epsilon}})]$$

$$\phi_{11}=U^{-1}\{\tilde{\phi}_{11}+\tfrac{1}{4}U^{-1}[-\tilde{\Delta}\tilde{\mathrm{D}}U-\bar{\tilde{\delta}}\tilde{\delta}U+\tfrac{3}{2}U^{-1}\,\tilde{\mathrm{D}}U\,\tilde{\Delta}U+\tfrac{3}{2}U^{-1}\,\tilde{\delta}U\,\bar{\tilde{\delta}}U-\bar{\tilde{\delta}}U\tilde{\tau}-\tilde{\delta}U\bar{\tilde{\tau}}$$
$$-\tilde{\Delta}U\tilde{\rho}+\tilde{\mathrm{D}}U\bar{\tilde{\mu}}+\tilde{\mathrm{D}}U(\tilde{\gamma}+\bar{\tilde{\gamma}})+\tilde{\delta}U(\tilde{\alpha}-\bar{\tilde{\beta}})]\}$$

$$\phi_{10}=U^{-1}(\tilde{\phi}_{10}-\tfrac{1}{2}U^{-1}\,\bar{\tilde{\delta}}\tilde{\mathrm{D}}U+\tfrac{3}{4}U^{-2}\,\bar{\tilde{\delta}}U\,\tilde{\mathrm{D}}U+\tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U\,\tilde{\alpha}+\tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U\bar{\tilde{\beta}}-\tfrac{1}{2}U^{-1}\,\tilde{\delta}U\bar{\tilde{\sigma}}$$
$$-\tfrac{1}{2}U^{-1}\,\bar{\tilde{\delta}}U\tilde{\rho})$$

$$\phi_{12}=U^{-1}[\tilde{\phi}_{12}-\tfrac{1}{2}U^{-1}\,\tilde{\delta}\tilde{\Delta}U+\tfrac{3}{4}U^{-2}\,\tilde{\delta}U\,\tilde{\Delta}U+\tfrac{1}{2}U^{-1}\,\bar{\tilde{\delta}}U\bar{\tilde{\lambda}}+\tfrac{1}{2}U^{-1}\,\tilde{\delta}U\tilde{\mu}$$
$$-\tfrac{1}{2}U^{-1}\,\tilde{\Delta}U(\bar{\tilde{\alpha}}+\tilde{\beta})]$$

$$\phi_{20} = U^{-1}[\tilde{\phi}_{20} - \tfrac{1}{2}U^{-1}\,\tilde{\bar{\delta}}\tilde{\bar{\delta}}U + \tfrac{3}{4}U^{-2}\,\tilde{\bar{\delta}}U\,\tilde{\bar{\delta}}U + \tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U\tilde{\lambda} - \tfrac{1}{2}U^{-1}\,\tilde{\Delta}U\tilde{\bar{\sigma}}$$
$$-\tfrac{1}{2}U^{-1}\,\tilde{\bar{\delta}}U(\tilde{\alpha} - \tilde{\bar{\beta}})]$$
$$\phi_{22} = U^{-1}[\tilde{\phi}_{22} - \tfrac{1}{2}U^{-1}\,\tilde{\Delta}\tilde{\Delta}U + \tfrac{3}{4}U^{-2}\,\tilde{\Delta}U\,\tilde{\Delta}U + \tfrac{1}{2}U^{-1}\,\tilde{\delta}U\tilde{\nu} + \tfrac{1}{2}U^{-1}\,\tilde{\bar{\delta}}U\tilde{\bar{\nu}}$$
$$-\tfrac{1}{2}U^{-1}\tilde{\Delta}U(\tilde{\gamma} + \tilde{\bar{\gamma}})]$$
$$2\Lambda = U^{-1}[2\tilde{\Lambda} - \tfrac{1}{2}U^{-1}\,\tilde{\bar{\delta}}\tilde{\delta}U + \tfrac{1}{2}U^{-1}\,\tilde{\Delta}\tilde{\mathrm{D}}U + \tfrac{1}{4}U^{-2}\,\tilde{\bar{\delta}}U\,\tilde{\delta}U - \tfrac{1}{4}U^{-2}\,\tilde{\Delta}U\,\tilde{\mathrm{D}}U + \tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U\tilde{\bar{\mu}}$$
$$-\tfrac{1}{2}U^{-1}\,\tilde{\Delta}U\tilde{\rho} + \tfrac{1}{2}U^{-1}\,\tilde{\bar{\delta}}U\tilde{\tau} - \tfrac{1}{2}U^{-1}\,\tilde{\delta}U(\tilde{\bar{\beta}} - \tilde{\alpha} - \tilde{\bar{\tau}}) - \tfrac{1}{2}U^{-1}\,\tilde{\mathrm{D}}U(\tilde{\gamma} + \tilde{\bar{\gamma}})].$$

The tetrad components of the Weyl tensor are related by $\psi_i = U^{-1}\tilde{\psi}_i$.

Note added in proof. It can be shown that the metrics corresponding to case 1 are NUT metrics and possess four Killing vectors, whereas the metrics corresponding to case 2 possess three Killing vectors. The Killing tensor associated with the separation in case 2 is found to be a symmetrical product of the three Killing vectors and so the separation is simply a 'disguised' trivial separation. In case 1 this is not so. The authors are indebted to a referee, Dr N M J Woodhouse, for pointing out this possibility.

References

Carter B 1968 *Commun. Math. Phys.* **10** 280–310
Dautcourt G, Papapetrou A and Treder H 1961 *Math. Nachr.* **23** 371
Dietz W 1976 *J. Phys. A: Math. Gen.* **9** 519–33
Kinnersley W 1969 *J. Math. Phys.* **10** 1195
Kundt W 1961 *Z. Phys.* **163** 77
Liouville J 1846 *J. Math., Paris* **11** 345
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 3
Petrov A Z 1969 *Einstein Spaces* (Oxford: Pergamon) § 51
Stäckel P 1893 *Math. Ann.* **42** 537
Woodhouse N M J 1975 *Commun. Math. Phys.* **44** 9–38